

Relationship Between Group Delay and Stored Energy in Microwave Filters

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Abstract—In this paper, an expression for the time average stored energy (t.a.s.e.) in a passive lossless two-port is derived in terms of its scattering parameters. In particular, it is shown that the t.a.s.e. in a passive lossless reciprocal symmetrical or antimetrical two-port is proportional to the group delay. One implication of this result is that the t.a.s.e., which is linked to the power-handling capability in many passive filters used in practice, is proportional to the group delay of the filter. This rigorous derivation is based on a variational theorem, which has been used in the past to prove energy storage results for passive lossless one-ports and periodic two-ports.

Index Terms—Group delay, lossless two-ports, microwave filters, power-handling capability, stored energy.

I. INTRODUCTION

THE power-handling capability of microwave filters is often limited by the breakdown of the dielectric inside the filter resonators. In his consideration of the peak internal fields in direct-coupled-cavity filters, Young [1], [2] sought to predict the peak internal fields in these filters in terms of the group delay of the filter. The basis of Young's approach was the assumption that the time average stored energy (t.a.s.e.) of the electromagnetic fields inside the filter could be expressed in terms of group delay. Young [1], [2] argued that in the passband and in the transition region adjacent to the passband edges, direct-coupled-cavity filters resembled periodic structures. Since the relationship between t.a.s.e. and group delay in periodic lossless two-ports was well established, Young assumed that the same relationship could be used for the filter. Young highlighted the limitations of his intuitive approach and pointed out that "...Of course, this will hold only to the extent that our direct-coupled-cavity filter resembles an infinite uniform periodic structure..."

In order to obtain a more solid foundation for the prediction of the power-handling capability of filters, the relationship between the t.a.s.e. in a passive lossless reciprocal two-port and its two-port scattering parameters is investigated. The relationships derived in this paper have been successfully used when comparing the power-handling capability of different filter realizations of the same power transfer function [3].

A rigorous derivation of the relationship between the t.a.s.e. in passive lossless two-ports and the scattering parameters is presented in this paper. In the case when the passive lossless two-port is reciprocal and symmetrical or antimetrical, it is

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shown that the average stored energy is proportional to the group delay. This covers most passive filters that are commonly used in practice, i.e., Butterworth, Chebyshev, and general Chebyshev filters. It also provides a justification for the assumption made by Young, and it shows that his results are not dependent on his argument that a direct-coupled-cavity filter resembles a periodic structure.

The derivation is based on the variational theorem used by Schwinger [4], who obtained an expression for the stored energy in a lossless two-port network in terms of its impedance matrix. Collin employed the variational theorem to establish the relationship between the t.a.s.e. and group delay in passive lossless *periodic* two-ports (see [5, p. 569]).

In order to demonstrate the validity of the equations derived in this paper, the energy stored in a simple example network is considered. Two cases are analyzed. In the first case, the network does not have any symmetry properties and, in the second case, it is symmetrical.

II. STORED ENERGY IN PASSIVE LOSSLESS TWO-PORTS

Assuming a harmonic variation of the electric field $\vec{E}(t)$ and the magnetic field $\vec{H}(t)$, the time average electric energy $W_{av,e}$ and the time average magnetic energy $W_{av,m}$ stored in a volume V filled with a lossless dispersive-free material, which is devoid of any energy sources, are given by [6]

$$W_{av,e} = \frac{1}{2} \iiint_V \epsilon \vec{E}(\omega) \cdot \vec{E}^*(\omega) dV \geq 0 \quad (1)$$

$$W_{av,m} = \frac{1}{2} \iiint_V \mu \vec{H}(\omega) \cdot \vec{H}^*(\omega) dV \geq 0 \quad (2)$$

where

$$\vec{E}(t) = \sqrt{2} \operatorname{Re} \left\{ \vec{E}(\omega) \cdot e^{j\omega t} \right\}, \quad \vec{E}(\omega) := \begin{bmatrix} \mathbf{E}_x(\omega) \\ \mathbf{E}_y(\omega) \\ \mathbf{E}_z(\omega) \end{bmatrix} \quad (3)$$

and

$$\vec{H}(t) = \sqrt{2} \operatorname{Re} \left\{ \vec{H}(\omega) \cdot e^{j\omega t} \right\}, \quad \vec{H}(\omega) := \begin{bmatrix} \mathbf{H}_x(\omega) \\ \mathbf{H}_y(\omega) \\ \mathbf{H}_z(\omega) \end{bmatrix}. \quad (4)$$

The components of \vec{E} and \vec{H} are complex quantities and are the rms-phasor representations of the fields in the x -, y -, and z -directions. The notation \vec{E}^* and \vec{H}^* indicates the complex conjugate of \vec{E} and \vec{H} , respectively.

The total t.a.s.e. $W_{av,tot}$ in the volume V is the sum of the time average stored electric and the time average stored magnetic

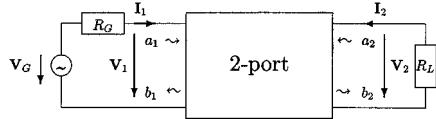


Fig. 1. Terminal voltages and currents and incident and reflected waves of a two-port.

energy, i.e.,

$$W_{\text{av, tot}} = \frac{1}{2} \iiint_V \left(\epsilon \vec{E} \cdot \vec{E}^* + \mu \vec{H} \cdot \vec{H}^* \right) dV \geq 0. \quad (5)$$

The above volume integral can be transformed into an integral over the closed surface of the volume using a variational theorem [4], [5], [7], i.e.,

$$W_{\text{av, tot}} = \frac{j}{2} \oint_S \left(\vec{E} \times \frac{d\vec{H}^*}{d\omega} + \frac{d\vec{E}^*}{d\omega} \times \vec{H} \right) \cdot \vec{u}_n ds \quad (6)$$

where \vec{u}_n is a unity vector normal to the closed surface S of the volume V and \vec{u}_n is directed into the volume. In the above, it has been assumed that the volume contains only lossless nondispersive materials. However, Collin has shown that (6) also holds when dispersive material is present in the volume V . In the case of a passive lossless two-port, (6) can be expressed as [4], [5]

$$W_{\text{av, tot}} = \frac{j}{2} \left[\oint_{S_1} \left(\vec{E}_1 \times \frac{d\vec{H}_1^*}{d\omega} + \frac{d\vec{E}_1^*}{d\omega} \times \vec{H}_1 \right) \cdot \vec{u}_n ds \right. \\ \left. + \oint_{S_2} \left(\vec{E}_2 \times \frac{d\vec{H}_2^*}{d\omega} + \frac{d\vec{E}_2^*}{d\omega} \times \vec{H}_2 \right) \cdot \vec{u}_n ds \right] \quad (7)$$

where S_1 and S_2 are the terminal surfaces correspondingly to ports 1 and 2, respectively. By applying Flouquet's theorem, it is possible to deduce from (7) that the t.a.s.e. in passive lossless *periodic* two-ports is proportional to the group delay (see [5, eq. (8.47b)]).

Following Schwinger [4], the surface integrals in (7) can then be expressed in terms of equivalent port rms-voltages \mathbf{V}_1 , \mathbf{V}_2 and equivalent port rms-currents \mathbf{I}_1 , \mathbf{I}_2 , as indicated in Fig. 1. Equation (7) then becomes

$$W_{\text{av, tot}} = \frac{j}{2} \left(\mathbf{V}_1 \frac{d\mathbf{I}_1^*}{d\omega} + \mathbf{I}_1 \frac{d\mathbf{V}_1^*}{d\omega} + \mathbf{V}_2 \frac{d\mathbf{I}_2^*}{d\omega} + \mathbf{I}_2 \frac{d\mathbf{V}_2^*}{d\omega} \right). \quad (8)$$

Equation (8) can be used to establish the relationship between stored energy and any set of two-port parameters.

Assuming real and frequency-independent reference impedances Z_1 and Z_2 at ports 1 and at 2, respectively, \mathbf{V}_1 , \mathbf{V}_2 , \mathbf{I}_1 , and \mathbf{I}_2 can be written in terms of power waves [8], i.e.,

$$\begin{aligned} \mathbf{V}_1 &= \sqrt{Z_1}(a_1 + b_1) \\ \mathbf{V}_2 &= \sqrt{Z_2}(a_2 + b_2), \\ \mathbf{I}_1 &= \frac{1}{\sqrt{Z_1}}(a_1 - b_1) \\ \mathbf{I}_2 &= \frac{1}{\sqrt{Z_2}}(a_2 - b_2) \end{aligned} \quad (9)$$

where the incident power waves a_1 and a_2 are related to the reflected power waves b_1 and b_2 by the scattering matrix $[s]$, i.e.,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \quad (10)$$

For a lossless network, the scattering matrix is unitary, i.e., the following conditions are satisfied regardless of the choice of the reference impedances:

$$|s_{11}|^2 + |s_{21}|^2 = 1 \quad (11)$$

$$|s_{22}|^2 + |s_{12}|^2 = 1 \quad (12)$$

$$s_{11}s_{12}^* + s_{21}s_{22}^* = 0 \quad (13)$$

$$s_{12}s_{11}^* + s_{22}s_{21}^* = 0. \quad (14)$$

Substituting (9) into (8) gives

$$W_{\text{av, tot}} = j \left(a_1 \frac{da_1^*}{d\omega} - b_1 \frac{db_1^*}{d\omega} + a_2 \frac{da_2^*}{d\omega} - b_2 \frac{db_2^*}{d\omega} \right) \quad (15)$$

and the t.a.s.e. in a passive lossless two-port is obtained in terms of incident and reflected power waves.

III. SINGLE GENERATOR AND MATCHED LOAD

Suppose that a generator with internal resistance R_G is connected to port 1 and that port 2 is terminated in its reference impedance Z_2 , as shown in Fig. 1. In this case

$$a_2 = 0 \quad (16)$$

and (10) reduces to

$$b_1 = s_{11}a_1 \quad (17)$$

$$b_2 = s_{21}a_1. \quad (18)$$

Substituting these equations into (15) and employing the unitary condition (11), the t.a.s.e. in the two-port is

$$W_{\text{av, tot}} = -j|a_1|^2 \left(s_{11} \frac{ds_{11}^*}{d\omega} + s_{21} \frac{ds_{21}^*}{d\omega} \right). \quad (19)$$

Equation (19) is valid for any passive lossless two-port. The case when a generator is connected to port 2 and port 1 is terminated in its reference impedance can be treated in an identical fashion to yield a result that can be obtained from (19) by simply interchanging the subscripts 1 and 2.

By expressing the scattering parameters in polar form, i.e., $s_{ij} = |s_{ij}| \cdot e^{j\phi_{ij}}$, and making use of the expression obtained differentiating (11), i.e.,

$$|s_{11}| \frac{d|s_{11}|}{d\omega} + |s_{21}| \frac{d|s_{21}|}{d\omega} = 0 \quad (20)$$

(19) can be simplified. When $0 < |s_{21}| < 1$, the incident power from the generator is neither totally reflected at the passive lossless two-port nor totally transmitted to the load and $0 < |s_{11}| < 1$. In this case, s_{11} and s_{21} can be expressed as $s_{11} = |s_{11}| \cdot e^{j\phi_{11}}$ and $s_{21} = |s_{21}| \cdot e^{j\phi_{21}}$. Since neither $|s_{11}|$

nor $|s_{21}|$ are zero, there is no problem in defining ϕ_{11} and ϕ_{21} and the t.a.s.e. in the two-port [see (19)] can be written as

$$W_{av, tot} = -j|a_1|^2 \left(|s_{11}| \frac{d|s_{11}|}{d\omega} - j|s_{11}|^2 \frac{d\phi_{11}}{d\omega} + |s_{21}| \frac{d|s_{21}|}{d\omega} - j|s_{21}|^2 \frac{d\phi_{21}}{d\omega} \right). \quad (21)$$

Substituting (20) into (21) and making use of (11) gives

$$W_{av, tot} = -|a_1|^2 \left([1 - |s_{21}|^2] \frac{d\phi_{11}}{d\omega} + |s_{21}|^2 \frac{d\phi_{21}}{d\omega} \right). \quad (22)$$

It is not possible to further reduce (22) unless some restrictions are placed on the passive lossless two-port. Hence, consider a passive lossless two-port, which is also reciprocal, i.e., $s_{21} = s_{12}$. The unitary conditions (11) and (12) then imply that

$$|s_{11}| = |s_{22}| \quad (23)$$

and a condition on the phases of s_{11} , s_{22} , and s_{21} can be obtained from (13) or (14)

$$|s_{11}||s_{21}| \left(e^{j(\phi_{11} - \phi_{21})} + e^{j(\phi_{21} - \phi_{22})} \right) = 0 \quad (24)$$

$$\Leftrightarrow \phi_{11} + \phi_{22} = 2\phi_{21} + (2k + 1)\pi, \quad k \in \mathbb{Z}. \quad (25)$$

Differentiating (25) yields

$$\frac{d\phi_{11}}{d\omega} + \frac{d\phi_{22}}{d\omega} = 2 \frac{d\phi_{21}}{d\omega}. \quad (26)$$

However, (26) is not sufficient to further reduce (22), but if additionally, the two phases ϕ_{11} and ϕ_{22} differ only by a constant Φ , i.e.,

$$\phi_{11} = \phi_{22} + \Phi, \quad \Phi \in \mathbb{R} \quad (27)$$

(26) gives

$$\frac{d\phi_{11}}{d\omega} = \frac{d\phi_{21}}{d\omega} \quad (28)$$

and, in this case, the t.a.s.e. is

$$W_{av, tot} = -|a_1|^2 \frac{d\phi_{21}}{d\omega} \quad (29)$$

after substituting (28) into (22). In this derivation, it has been necessary to assume that $0 < |s_{21}| < 1$. It remains to consider Cases (I) $|s_{21}| = 1$ and (II) $|s_{21}| = 0$.

A. Case (I): $|s_{21}| = 1$

When $|s_{21}| = 1$, all of the incident power from the generator reaches the load, and from (11), $s_{11} = 0$, and from (20)

$$\frac{d|s_{21}|}{d\omega} = 0. \quad (30)$$

In this case, (19) reduces to

$$W_{av, tot} = -|a_1|^2 \frac{d\phi_{21}}{d\omega} \quad (31)$$

where ϕ_{21} is the phase of s_{21} . This is the same as (29), but nothing other than a passive lossless two-port has been assumed. It corresponds, for example, to a transmission line terminated in a matched load and excited by a matched generator. In this case, the quantity $|a_1|^2$ is the available power of the generator and $-(d\phi_{21}/d\omega)$ is the associated group delay of the transmission line.

B. Case (II): $|s_{21}| = 0$

When $|s_{21}| = 0$, i.e., $s_{21} = 0$, none of the incident power from the generator reaches the load, and from (11), $|s_{11}| = 1$ and from (20)

$$\frac{d|s_{11}|}{d\omega} = 0. \quad (32)$$

In this case, (19) reduces to

$$W_{av, tot} = -|a_1|^2 \frac{d\phi_{11}}{d\omega}. \quad (33)$$

This equation, for example, is applicable when the two-port consists of two isolated passive lossless lumped one-ports, and also at frequencies at which a filter exhibits transmission zeros.

Note, that in (19), (21), (22), (29), (31), and (33), it is assumed that the load resistance R_L equals the reference impedance of port 2. Of practical interest is the case when the reference impedances Z_1 and Z_2 are chosen to be the terminating generator and load resistances, respectively. The quantity $|a_1|^2$ is then the available power P_A of the generator and $-(d\phi_{21}/d\omega)$ in (29) and (31) is the associated group delay T_g of the two-port.

IV. STORED ENERGY IN A PASSIVE LOSSLESS RECIPROCAL SYMMETRICAL OR ANTSYMMETRICAL TWO-PORT

Suppose a reciprocal two-port is terminated in a matched load R_L and excited by a matched generator with an internal resistance R_G identical to the load resistance, i.e., $R_L = R_G = R_0$. This implies that the two reference impedances Z_1 and Z_2 are identical and equal to R_0 . Hence, if the two-port is also symmetrical, the scattering parameters s_{11} and s_{22} are related to each other by

$$s_{11} = s_{22}. \quad (34)$$

Similarly, for a passive lossless reciprocal antimetrical network

$$s_{11} = -s_{22}. \quad (35)$$

In both cases, (27) is satisfied and the t.a.s.e. in a passive lossless reciprocal symmetrical or antimetrical network is given by (29) when $0 < |s_{21}| \leq 1$ and (33) when $|s_{21}| = 0$. Equation (29) has the form

$$W_{av, tot} = P_A \cdot T_g. \quad (36)$$

where P_A is the available power from the generator and $T_g = -(d\phi_{21}/d\omega)$ is the group delay.

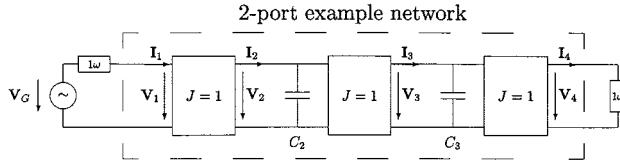


Fig. 2. Example two-port to verify the derived equations.

V. EXAMPLES

Consider the example network in Fig. 2, the scattering matrix of which is given by

$$s_{11} = \frac{\omega(C_2 - C_3) + j\omega^2 C_2 C_3}{\omega(C_2 + C_3) + j(\omega^2 C_2 C_3 - 2)} \quad (37)$$

$$s_{22} = \frac{\omega(C_3 - C_2) + j\omega^2 C_2 C_3}{\omega(C_2 + C_3) + j(\omega^2 C_2 C_3 - 2)} \quad (38)$$

$$s_{12} = \frac{2}{\omega(C_2 + C_3) + j(\omega^2 C_2 C_3 - 2)} = s_{21} \quad (39)$$

with respect to 1Ω reference impedances.

There is no energy stored in the admittance inverters. Thus, the overall energy stored in this lossless reciprocal two-port may be obtained by computing the voltages across the two capacitors C_2 and C_3 and then adding up their stored energies

$$W_{Cr} = |\mathbf{V}_r|^2 \frac{C_r}{2}, \quad r = 2, 3 \quad (40)$$

giving

$$W = \frac{|\mathbf{V}_G|^2}{2} \frac{\omega^2 C_2 C_3 + C_2 + C_3}{\omega^4 C_2^2 C_3^2 + \omega^2 (C_3 - C_2)^2 + 4}. \quad (41)$$

Alternatively, the stored energy can be calculated using the equations derived in this paper given the scattering matrix of the two-port.

A. Symmetrical Network

Suppose the network in Fig. 2 is symmetrical, i.e., $C_2 = C_3 = C$. The stored energy can then be computed from the phase of the transmission coefficient

$$\phi_{12} = -\arctan \frac{\omega^2 C^2 - 2}{2\omega C} \pm \pi \quad (42)$$

using (36), i.e.,

$$W = -|a_1|^2 \frac{d\phi_{12}}{d\omega} = |\mathbf{V}_G|^2 \frac{C}{2} \frac{\omega^2 C^2 + 1}{\omega^4 C^4 + 4} \quad (43)$$

and $-(d\phi_{12}/d\omega)$ is the group delay. This agrees with the result obtained when (41) is used.

B. Nonsymmetrical Network

If, however, the network does not have any symmetry properties, i.e., $C_2 \neq C_3$, in this case, (36) gives

$$-|a_1|^2 \frac{d\phi_{12}}{d\omega} = \frac{|\mathbf{V}_G|^2}{4} \frac{(C_2 + C_3)(\omega^2 C_2 C_3 + 2)}{\omega^4 C_2^2 C_3^2 + \omega^2 (C_3 - C_2)^2 + 4}. \quad (44)$$

It is seen that this does not yield the stored energy in the network. Equations (19) or (22) must be used instead to obtain the stored

energy in the example network, and it has been verified using Maple V that this leads to the same result as in (41).

C. Frequency Transformations

Recently, [3] (36) has also been used to relate the stored energy W_{LP} in symmetrical low-pass filters to the stored energy W_{BP} in bandpass filters obtained by applying the low-pass to bandpass transformation [9]

$$T: \omega \mapsto \alpha \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right). \quad (45)$$

Employing (36), it can be shown that the stored energy in the derived bandpass filter is given by

$$W_{BP} = W_{LP}(T(\omega)) \cdot \frac{dT(\omega)}{d\omega}. \quad (46)$$

This can, of course, be generalized to any other frequency transformation. Predictions of the energy storage and, hence, the power-handling capability in filters, can be reduced to the consideration of the low-pass prototype from which it is derived.

VI. CONCLUSION

A rigorous derivation of the relationship between the total t.a.s.e. in a passive lossless two-port and its scattering parameters has been presented in this paper.

In particular, it is shown that the t.a.s.e. in a passive lossless reciprocal symmetrical or antimetrical two-port is proportional to the group delay. Hence, it can be concluded that the t.a.s.e. in a wide-range of commonly used passive filters is, in practice, proportional to the group delay of the filter. This interesting result can be expected to lead to useful insight into the power-handling capability of these types of filters, since the electromagnetic fields, which can cause the dielectric breakdown in a filter, are directly related to the t.a.s.e. in a filter.

The result also shows how the t.a.s.e. in a filter is related to the t.a.s.e. of the low-pass prototype from which it is derived. Hence, a prediction of the maximum field in a filter can be obtained from the analysis of the low-pass prototype [3].

The direct-coupled-cavity filters considered by Young are also lossless reciprocal symmetrical two-ports driven by a matched generator and terminated in a matched load. Hence, the group delay is proportional to the t.a.s.e. in these filters. This agrees with Young's intuitively derived relationship between the t.a.s.e. and group delay [1], [2] for frequencies in the passband.

This paper can be extended to the consideration of passive lossless n -ports ($n > 2$) connected to m generators ($1 \leq m \leq n$).

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